Role of Zero Modes in Quantization of QCD in Light-Cone Coordinates

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Abstract

Two-dimensional heavy-quark QCD is studied in the light-cone coordinates with (anti-) periodic field boundary conditions. We carry out the light-cone quantization of this gauge invariant model. To examine the role of the zero modes of the gauge degrees of freedom, we consider the quantization procedure in the zero mode and the nonzero mode sectors separately. In both sectors, we obtain the physical variables and their canonical (anti-) commutation relations. The physical Hamiltonian is constructed via a step-by-step elimination of the unphysical degrees of freedom. It is shown that the zero modes play a crucial role in the self-interaction potential of both the heavy-quarks and gluons, and in the interaction potential between them. It is also shown that the Faddeev-Popov determinant depends on the zero modes of the gauge degrees of freedom. Therefore, one needs to introduce the Faddeev-Popov ghosts in their own nonzero mode sector.

1 Introduction

One of the promising approaches to problems in QCD is the light-cone quantization [1, 2]. The light-cone quantization has turned out to be a useful tool for the perturbative treatment of field theories [3, 4]. In its extension [9-13] to the nonperturbative domain, one has come to realize that careful attention must be paid to the nontrivial vacuum structure of the light-cone quantum field theory. Some mathematical aspects of the question with regard to the existence of such vacuum states were considered in [10, 11]. For instance, the light-cone vacuum in the massless Schwinger model can be only understood by careful study of the zero modes of the constraints imposed by the light-cone frame [12, 13]. Indeed, it has been conjectured that the dynamics of the zero modes in QCD in the light-cone quantization provides the mechanism for the confinement [1, 2].

In the present paper, we continue the quantization of heavy-fermion gauge theory started in our previous article [14]. We apply the Faddeev-Jackiw quantization technique to the two-dimensional heavy-fermion QED and two-dimensional heavy-quark QCD. Although in this case the quantization of the full model (not only in heavy mass limit but also in light mass limit) can be performed, we restrict ourselves to heavy-fermions. We carry out the quantization of these models in a light-cone domain restricted in its "spatial" directions. It is well known that in such a restricted region one has problems with the zero modes [15, 13] which, as it was mentioned above, turned out to be the most important variables in this case. The quantization of QED and QCD on the finite dimension manifolds (circle, torus) were considered in [16, 17, 18]. The role of the zero modes in QED using the Lagrange approach was studied in [19]. The dynamics of zero modes in the two-dimensional QCD employing the light-cone variables was considered in [20, 21, 22, 23].

In studying the quantization of the gauge field theories, one is confronted by first-class constraints, and, for this reason, the corresponding gauge conditions should be imposed. A consistent canonical quantization formalism for such problems was proposed by Dirac [24] and Bergmann [25], and its generalization to fermionic (Grassmann-odd) constraints by Casalbuoni [26]. There is another approach to the quantization of the gauge theories proposed by Faddeev and Jackiw [27].

The specific gauge theories (QED and QCD) we address in this paper are in terms of the light-cone variables where, as we will see, the quantization faces with some constraints involving the zero mode variables which require special attention. To examine explicitly the role of the zero modes, we consider the quantization procedure in the separated nonzero mode and zero mode sectors. In such sectors, we choose special gauge conditions in order to gauge out the nonzero modes and obtain the physical variables and their canonical (anti-) commutation relations. The physical Hamiltonian is constructed by systematic elimination of the nondynamical variables.

The paper is organized as follows. In Section 2, the Faddeev-Jackiw technique in terms of the light-cone coordinates is applied to the two-dimensional heavy-fermion QED. Despite that the results we obtain here are trivial, this example is useful for better understanding what is going on in QCD. The Faddeev-Jackiw quantization algorithm is equivalent to the Dirac one, of course, but it is sometimes simpler to be employed, especially when the constraints are linear. In Section 3, we consider the light-cone quantization of the two-dimensional heavy-quark QCD, where the zero modes and nonzero modes of the gauge degrees of freedom are taken into account. It is shown that the physical degrees of freedom are the zero modes of gauge fields ($A_-^{(P)}$ in QED and $A_-^{a(P)}$ in QCD), their conjugate momenta and the fermionic variables. We found that the careful elimination of unphysical gauge degrees of freedom leads to additional terms in the physical Hamiltonian. Such a Hamiltonian is constructed, and, in particular, the interaction potential between heavy-quarks, as well as the interaction Hamiltonian between heavy-quarks and gluons are found. It is shown that in the case of QCD one needs to introduce the nonzero modes of the Faddeev-Popov ghosts.

2 2D Heavy Fermion QED

In this Section, we are going to consider a simple example which illustrates the Faddeev-Jackiw quantization technique [27] for the case of the 2D heavy-fermion QED when one needs to take the zero modes of the gauge degrees of freedom into consideration. This will be the first step toward the attack of the QCD model, so we can kill two birds by one stone.

Following the Isgur and Wise [28], the Lagrange density of the $2D$ heavy-fermion QED has the following form

$$\mathcal{L} = i\overline{\Psi}\not{U}U^{\mu}D_{\mu}\Psi - M\overline{\Psi}\Psi + \frac{1}{2}F_{03}^2 \tag{2.1}$$

where the Minkowski metric is: diag $g_{\mu\nu} = (1, -1)$,

$$D_{\mu} = \partial_{\mu} + ieA_{\mu} \tag{2.2}$$

is the covariant derivative,

$$\not{U} = \gamma \cdot U = \gamma^{\mu}U_{\mu} \tag{2.3}$$

and U^{μ} is a given 2-velocity of the heavy-fermions subject to the condition $U^2 = 1$. The field strength tensor is

$$F_{03} = \partial_0 A_3 - \partial_3 A_0 \tag{2.4}$$

and we use the system of units where $\hbar = c = 1$. The heavy-fermion limit means that the quantity MU^{μ} is greater than any other momenta in the problem under consideration.

The light-cone coordinates in the two-dimensional space are $x^{\mu} = (x^+, x^-)$, where

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \tag{2.5}$$

The variable x^+ plays the role of the "time" variable, and x^- is the spatial one. In terms of such coordinates, the Lagrange density $\mathcal{L}$ becomes

$$\begin{aligned}\mathcal{L} &= i\overline{\Psi}\not{U}(U_+\partial_- + U_-\partial_+)\Psi - M\overline{\Psi}\Psi + \frac{1}{2}F_{+-}^2 \\ &\quad -e\overline{\Psi}\not{U}(U_+A_- + U_-A_+)\Psi\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}F_{+-} &= \partial_+A_- - \partial_-A_+ \ , \ A_{\pm} = \frac{1}{\sqrt{2}}(A_0 \pm A_3) \ , \\ \partial_{\pm} &= \frac{\partial}{\partial x^{\pm}} \ , \ U_{\pm} = \frac{1}{\sqrt{2}}(U_0 \pm U_3)\end{aligned} \tag{2.7}$$

The Lagrange density (2.1) is gauge invariant. This means that the classical theory contains "first-class" constraints, and one needs a quantization prescription for systems with constraints such as that provided, for example, by Dirac [24], Bergmann [25], Casalbuoni [26] or Faddeev-Jackiw [27].

The infrared problems can be regularized by considering the system to be contained in a finite volume. We consider, for this reason, the quantization of the theory in the restricted region

$-L \leq x^- \leq L$, and impose periodic boundary conditions for the bosonic variables and antiperiodic boundary conditions for the fermionic ones

$$
\begin{aligned}
&A_\mu(x^+, x^- - L) = A_\mu(x^+, x^- + L), \\
&\Psi(x^+, x^- - L) = -\Psi(x^+, x^- + L), \quad \bar{\Psi}(x^+, x^- - L) = -\bar{\Psi}(x^+, x^- + L)
\end{aligned} \tag{2.8}
$$

We choose the antiperiodic boundary conditions for the fermions in order to avoid their own zero mode problem, and treat them as Grassmann-odd variables. The periodic boundary conditions for the gauge variables are useful because when integrating by parts the boundary term vanishes. Although we choose antiperiodic boundary conditions for the fermions, this does not have to confuse anyone because the fermions always appear in a bilinear combination which is a periodic function. Each boundary condition implies that there is an additional constraint to be satisfied. In this paper, the periodic constraints are imposed explicitly. In [29] the boundary conditions were treated as additional constraints.

To start with quantization procedure, we define the momenta Π_Ψ and $\Pi_{\overline{\Psi}}$ conjugate to the fermionic variables Ψ and $\overline{\Psi}$, and the momenta $\Pi_\pm$ conjugate to the gauge degrees of freedom $A_\pm$

$$
\begin{aligned}
&\Pi_\Psi = \frac{\partial_r \mathcal{L}}{\partial \dot{\Psi}} = i\overline{\Psi}\not{U}U_-, \quad \Pi_{\overline{\Psi}} = \frac{\partial_r \mathcal{L}}{\partial \dot{\overline{\Psi}}} = 0 \;, \\
&\Pi_\pm = \frac{\partial \mathcal{L}}{\partial \dot{A}_\pm}
\end{aligned} \tag{2.9}
$$

where the label "r" denotes the right derivative, and the "dot" always means the derivative with respect to the "time" x^+.

The zero modes and nonzero modes for the gauge degrees of freedom are defined as

$$
A_\pm^{(P)} = P * A_\pm \;\;,\;\; A_\pm^{(Q)} = Q * A_\pm \tag{2.10}
$$

where P and Q are the projection operators onto the zero mode sector (P sector) and nonzero mode sector (Q sector), respectively [13]

$$
\begin{aligned}
P(x,y) &= \frac{1}{2L} \;\;,\;\; Q(x,y) = \delta(x^- - y^-) - P(x,y) \;, \\
(P * f)(x) &= \frac{1}{2L}\int_{-L}^{L} f(x)dx
\end{aligned} \tag{2.11}
$$

The momenta

$$
\Pi_\pm^{(P,Q)} \equiv \left(\Pi^{(P,Q)}_{A_\pm^{(P,Q)}}\right)
$$

conjugate to the variables $A_\pm^{(P,Q)}$ are

$$
\Pi_\pm^{(P)} = P * \frac{\partial \mathcal{L}}{\partial \dot{A}_\pm} \;\;,\;\; \Pi_\pm^{(Q)} = Q * \frac{\partial \mathcal{L}}{\partial \dot{A}_\pm} \tag{2.12}
$$

One then obtains

$$
\Pi_+^{(P,Q)} = 0 \;\;,\;\; \Pi_-^{(P)} = \dot{A}_-^{(P)} \;\;,\;\; \Pi_-^{(Q)} = \dot{A}_-^{(Q)} - \partial_- A_+^{(Q)} \tag{2.13}
$$

The velocities $\dot{A}_-^{(P,Q)}$ can be expressed through the momenta $\Pi_-^{(P,Q)}$, whereas $\dot{A}_+^{(P,Q)}$ can not. Consequently, one has four primary constraints $\mathbf{\Phi} = 0$

$$\mathbf{\Phi} = \left\{ \begin{array}{l} \phi_+^{(P,Q)} = \Pi_+^{(P,Q)} \\ \phi_\Psi = \Pi_\Psi - i\overline{\Psi}\not{U}U_- \\ \phi_{\overline{\Psi}} = \Pi_{\overline{\Psi}} \end{array} \right. \tag{2.14}$$

In terms of the variables $A_-^{(P,Q)}$, $\Pi_-^{(P,Q)}$, Π_Ψ, and Ψ, the Lagrange density $\mathcal{L}$ on the constraints surface (2.14) can be rewritten in the Hamilton form

$$\begin{aligned} \mathcal{L} &= \Pi_-^{(P)}\dot{A}_-^{(P)} + \Pi_-^{(Q)}\dot{A}_-^{(Q)} + \Pi_\Psi\dot{\Psi} - \mathcal{H}, \\ \mathcal{H} &= \mathcal{H}_F + \mathcal{H}_{EM}, \\ \mathcal{H}_F &= -\frac{U_+}{U_-}\Pi_\Psi\partial_-\Psi - i\frac{M}{U_-}\Pi_\Psi\not{U}\Psi \ , \\ \mathcal{H}_{EM} &= -ie\Pi_\Psi U_-^{-1}\Psi\left\{U_-A_+^{(P)} + U_-A_+^{(Q)} + U_+A_-^{(P)} + U_+A_-^{(Q)}\right\} \\ &\quad + \frac{1}{2}\left(\Pi_-^{(P)}\right)^2 + \frac{1}{2}\left(\Pi_-^{(Q)}\right)^2 + \Pi_-^{(Q)}\partial_-A_+^{(Q)} \end{aligned} \tag{2.15}$$

We neglect the term $\Pi_-^{(P)}\Pi_-^{(Q)}$ in the Hamilton density $\mathcal{H}_{EM}$ because it does not give a contribution to the corresponding Hamiltonian.

We now start the quantization procedure following [27]. (We do not present here the details of this algorithm. The readers can find them in the original paper [27]). To start with, we should define the initial set of variables $\zeta_j^{(0)}$ and the corresponding "generalized" momenta $a_j^{(0)}(\zeta^{(0)})$ which, in the case under consideration, are found to be

$$\begin{aligned} \zeta_j^{(0)} &= \left\{A_-^{(P)} \ , \ \Pi_-^{(P)} \ , \ A_+^{(P)} \ , \ A_-^{(Q)} \ , \ \Pi_-^{(Q)} \ , \ A_+^{(Q)} \ , \ \Psi \ , \ \Pi_\Psi\right\}, \\ a_j^{(0)} &= \left\{\zeta_2^{(0)} \ , \ 0 \ , \ 0 \ , \ \zeta_5^{(0)} \ , \ 0 \ , \ 0 \ , \ \zeta_8^{(0)} \ , \ 0\right\} \quad , \quad j = 1, \ldots, 8 \end{aligned} \tag{2.16}$$

One of the most important objects in the Faddeev-Jackiw approach is the simplectic supermatrix $f_{ij}^{(0)}$, which is defined in general by [30]

$$f_{ij}^{(0)} = \frac{\partial_\ell a_j^{(0)}}{\partial\zeta_i^{(0)}} - (-1)^{\epsilon_a\epsilon_\zeta}\frac{\partial_\ell a_i^{(0)}}{\partial\zeta_j^{(0)}} \tag{2.17}$$

where $\epsilon_a(\epsilon_\zeta)$ is the Grassmann parity of $a(\zeta)$

$$\epsilon_\zeta = \left\{ \begin{array}{l} 0 \ , \ \text{if} \ \zeta \ \text{is Grassmann} - \text{even} \\ 1 \ , \ \text{if} \ \zeta \ \text{is Grassmann} - \text{odd} \end{array} \right. \tag{2.18}$$

The simplectic supermatrix corresponding to the set of the variables (2.16) is block-diagonal

$$\text{block diag } f_{ij}^{(0)}(x^-, y^-) = (\mathcal{A}, 0, \mathcal{A}\delta(x^- - y^-), 0, \mathcal{A}_1\delta(x^- - y^-)) \tag{2.19}$$

where

$$\mathcal{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \mathcal{A}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.20}$$

The appropriate supermatrix $f^{(0)}_{ij}$ has two eigenvectors with zero eigenvalue ($f^{(0)}_{ij} V_j = 0$)

$$
\begin{aligned}
&V^{(P)} = \{0\,,\ 0\,,\ c\,, \dots\, 0\}\ , \\
&V^{(Q)} = \{0\,,\ \dots\ ,\, c(x^-)\,,\ 0\,,\ 0\}
\end{aligned} \tag{2.21}
$$

where c is a constant, and $c(x^-)$ is a function. This leads to other two constraints

$$
(\Pi_\Psi \Psi)^{(P)} = 0\ , \tag{2.22}
$$

$$
\partial_- \Pi_-^{(Q)} + ie\,(\Pi_\Psi \Psi)^{(Q)} = 0 \tag{2.23}
$$

After the first reduction procedure, the Lagrange density becomes

$$
\begin{aligned}
\mathcal{L}^{(1)} &= \Pi_-^{(P)} \dot{A}_-^{(P)} + \Pi_-^{(Q)} \dot{A}_-^{(Q)} + \Pi_\Psi \dot{\Psi} + \dot{\lambda} \left(\partial_- \Pi_-^{(Q)} + ie\,(\Pi_\Psi \Psi)^{(Q)} \right) \\
&\quad - \mathcal{H}_F + ie\,(\Pi_\Psi \Psi)^{(Q)}\, U_+ U_-^{-1} A_-^{(Q)} - \frac{1}{2}\left(\Pi_-^{(P)}\right)^2 - \frac{1}{2}\left(\Pi_-^{(Q)}\right)^2
\end{aligned} \tag{2.24}
$$

where λ is a Lagrange multiplier for the constraint (2.23).

The constraint (2.22) is of the first class (in Dirac classification). Therefore one needs a corresponding gauge condition. Instead of this, the constraint (2.22) is to be satisfied on the physical state vector. As a result, one can consider the fermionic degrees of freedom, Π_Ψ and Ψ, as independent ones. Using the constraints (2.23), one can express, in this case, the momentum $\Pi_-^{(Q)}$ in terms of the fermionic variables

$$
\Pi_-^{(Q)} = -ie\partial_-^{-1}\,(\Pi_\Psi \Psi)^{(Q)} \tag{2.25}
$$

where ∂_-^{-1} is the operator inverse to ∂_-, and whose matrix elements in the coordinate representation (in the Q sector) are

$$
G^{(Q)}(x^- - y^-) = \frac{\epsilon(x^- - y^-)}{2} - \frac{x^- - y^-}{2L} \tag{2.26}
$$

(One has to keep in mind that the operator ∂_-^{-1} is defined only on the Q sector space. Therefore (2.25) represents the only solution to the constraint equation (2.23).)

The reduced set of variables now is

$$
\begin{aligned}
\zeta_j^{(1)} &= \left\{ A_-^{(P)}\,,\ \Pi_-^{(P)}\,,\ A_-^{(Q)}\,,\ \Pi_-^{(Q)}\,,\ \lambda\,,\ \Psi\,,\ \Pi_\Psi \right\}\ ,\ j = 1, \dots, 7\ , \\
a_j^{(1)} &= \left\{ \zeta_2^{(1)}\,,\ 0\,,\ \zeta_4^{(1)}\,,\ 0\,,\ \partial_- \zeta_4^{(1)} + ie(\zeta_7^{(1)} \zeta_6^{(1)})^{(Q)}\,,\ \zeta_7^{(1)}\,,\ 0 \right\}
\end{aligned} \tag{2.27}
$$

The corresponding simplectic supermatrix $f^{(1)}_{ij}$ is still singular, but now it has only one eigenvector V with zero eigenvalue

$$
V = \{0\,,\, 0\,,\ -\partial_- f(x^-)\,,\ 0\,,\ f(x^-)\,,\ -ie\Psi f(x^-)\,,\ ie\Pi_\Psi f(x^-)\} \tag{2.28}
$$

where $f(x^-)$ is a function of x^-. This vector does not give any new constraints. Consequently, according to [27], one needs a gauge condition.

In the Faddeev-Jackiw quantization procedure, there is only one restriction on how one chooses gauge conditions: after imposing gauge conditions, the appropriate simplectic supermatrix should be nonsingular. On the other hand, gauge conditions should eliminate the gauge freedom.

Let us consider one of the possibilities of how one can choose the gauge condition in the problem considered. Using the constraint (2.23), the term

$$-\, ie\frac{U_+}{U_-}\,(\Pi_\Psi\Psi)^{(Q)}\,A_-^{(Q)} \tag{2.29}$$

in the Lagrange density (2.24) can be rewritten as

$$-\frac{U_+}{U_-}\Pi_-^{(Q)}\partial_- A_-^{(Q)} + \frac{U_+}{U_-}\partial_-\left(\Pi_-^{(Q)}A_-^{(Q)}\right) \tag{2.30}$$

A natural gauge condition is chosen to be (see [14])

$$\Omega_G = \partial_- A_-^{(Q)} = 0 \tag{2.31}$$

From (2.31), it follows that the nonzero mode $A_-^{(Q)}$ is an arbitrary function of x^+. On the other hand, the nonzero mode cannot depend only on x^+, otherwise it is the zero mode by definition. Therefore, the only solution to the eq.(2.31) is

$$A_-^{(Q)} = 0 \ \ (\text{note that } A_- \neq 0) \tag{2.32}$$

According to (2.31) and the boundary conditions we have chosen, the term (2.30) does not give a contribution to the Hamiltonian, and can be neglected in the Hamilton density.

The Lagrange density after the second reduction is

$$\begin{aligned}
\mathcal{L}^{(2)} &= \Pi_-^{(P)}\dot{A}_-^{(P)} + \Pi_-^{(Q)}\dot{A}_-^{(Q)} + \Pi_\Psi\dot{\Psi} + \dot{\lambda}\left(\partial_-\Pi_-^{(Q)} + ie\,(\Pi_\Psi\Psi)^{(Q)}\right) \\
&\quad + \dot{\beta}\partial_- A_-^{(Q)} - \mathcal{H}^{(2)}\,, \\
\mathcal{H}^{(2)} &= \frac{1}{2}(\Pi_-^{(P)})^2 + \frac{1}{2}(\Pi_-^{(Q)})^2 + \mathcal{H}_F
\end{aligned} \tag{2.33}$$

where β is the Lagrange multiplier for the gauge condition (2.31). Now the sets of variables are

$$\begin{aligned}
\zeta_k^{(2)} &= \left\{A_-^{(P)}\,,\ \Pi_-^{(P)}\,,\ A_-^{(Q)}\,,\ \Pi_-^{(Q)}\,,\ \lambda\,,\ \beta\,,\ \Psi\,,\ \Pi_\Psi\right\}\,,\ k = 1,\ldots,8\,, \\
a_k^{(2)} &= \left\{\zeta_2^{(2)}\,,\ 0\,,\ \zeta_4^{(2)}\,,\ 0\,,\ \partial_-\zeta_4^{(2)} + ie(\zeta_8^{(2)}\zeta_7^{(2)})^{(Q)}\,,\ \partial_-\zeta_3^{(2)}\,,\ \zeta_8^{(2)}\,,\ 0\right\}
\end{aligned} \tag{2.34}$$

and the corresponding simplectic supermatrix has the following block-diagonal form

$$f_{jk}^{(2)} = \begin{pmatrix} \mathcal{A} & \mathcal{O} \\ \mathcal{O}^T & \mathcal{B}(x^-)\delta(x^- - y^-) \end{pmatrix} \tag{2.35}$$

where

$$\mathcal{B}(x^-) = \begin{pmatrix}
0 & -1 & 0 & \partial_- & 0 & 0 \\
1 & 0 & \partial_- & 0 & 0 & 0 \\
0 & -\partial_- & 0 & 0 & ie\Pi_\Psi & -ie\Psi \\
-\partial_- & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -ie\Pi_\Psi & 0 & 0 & 1 \\
0 & 0 & ie\Psi & 0 & 1 & 0
\end{pmatrix} \tag{2.36}$$

the matrix $\mathcal{A}$ was defined in (2.20), the matrix $\mathcal{O}$ is a zero rectangular 2×6 matrix, and the symbol T stands for transposition.

The simplectic supermatrix $f^{(2)}_{jk}$ is nonsingular, and can be inverted using the Berezin algorithm [31] for the inversion of a supermatrix. The result is found to be

$$\left(f^{(2)}\right)^{-1}_{jk} = \begin{pmatrix} -\frac{1}{2L}\mathcal{A} & \mathcal{O} \\ \mathcal{O}^T & \mathcal{B}^{-1}(x^-)\delta(x^- - y^-) \end{pmatrix} \tag{2.37}$$

where

$$\mathcal{B}^{-1}(x^-) = \begin{pmatrix} 0 & 0 & 0 & -\partial_-^{-1} & 0 & 0 \\ 0 & 0 & -\partial_-^{-1} & 0 & -ie\partial_-^{-1}\Psi & ie\partial_-^{-1}\Pi_\Psi \\ 0 & \partial_-^{-1} & 0 & \partial_-^{-2} & 0 & 0 \\ \partial_-^{-1} & 0 & -\partial_-^{-2} & 0 & -ie\partial_-^{-2}\Psi & -ie\partial_-^{-2}\Pi_\Psi \\ 0 & -ie\Psi\partial_-^{-1} & 0 & -ie\Psi\partial_-^{-2} & 0 & 1 \\ 0 & ie\Pi_\Psi\partial_-^{-1} & 0 & ie\Pi_\Psi\partial_-^{-2} & 1 & 0 \end{pmatrix} \tag{2.38}$$

The operator ∂_-^{-2} is the one whose matrix elements in the coordinate representation (in the Q sector) are

$$H^{(Q)}(x^- - y^-) \; = \; \frac{|x^- - y^-|}{2} - \frac{(x^- - y^-)^2}{4L} - \frac{2L}{3} \tag{2.39}$$

The nonsingularity of the supermatrix (2.37) is consistent with the gauge condition (2.31) we have chosen.

The structure of the supermatrix (2.37) shows that the physical variables are the fermionic ones, Π_Ψ and Ψ, plus the zero modes of Π_- and A_-, thereby

$$\omega^{\rm phys} = \{A_-^{(P)} \;,\; \Pi_-^{(P)} \;,\; \Psi \;,\; \Pi_\Psi\} \tag{2.40}$$

satisfying the following brackets

$$\begin{aligned} \left\{A_-^{(P)} \;,\; \Pi_-^{(P)}\right\}_{FJ} &= \frac{1}{2L}, \\ \{\Psi(x^-) \;,\; \Pi_\Psi(y^-)\}_{FJ} &= \delta(x^- - y^-) \end{aligned} \tag{2.41}$$

These brackets coincide with those obtained in [14] using the Dirac method of quantization for the systems with “first-class” constraints.

The quantization procedure consists of the replacement of the variables $\omega^{\rm phys}$ by the corresponding operators

$$\omega^{\rm phys} \to \hat{\omega}^{\rm phys} \tag{2.42}$$

which obey the following commutation relations

$$\begin{aligned} &[\hat{A}_-^{(P)},\, \hat{\Pi}_-^{(P)}]_- = \frac{i}{2L}, \\ &[\hat{\Psi}(x^-),\, \hat{\Pi}_\Psi(y^-)]_+ = i\delta(x^- - y^-) \end{aligned} \tag{2.43}$$

The Hamilton density corresponding to the physical Hamiltonian can be found by solving the constraints (2.23), which is equivalent to the substitution of $\Pi_-^{(Q)}$ (2.25) into (2.33). This gives

$$\mathcal{H}^{\rm phys} = \frac{1}{2}\left(\Pi_-^{(P)}\right)^2 + \frac{e^2}{2}\left(\partial_-^{-1}\left(Q * \overline{\Psi}\not{U}_-\Psi\right)\right)^2 + \mathcal{H}_F \tag{2.44}$$

It should be mentioned that, in the $2D$ QED with zero modes taken into account, there is no real interaction between the gauge field and heavy-fermions.

Let us discuss the constraint (2.22). As it was mentioned above, it is impossible (at least we do not know how to do this) to fix the gauge corresponding to this constraint. We will consider, for this reason, the constraint (2.22) as a strong one, meaning that this constraint should be satisfied on the physical state vectors $|\mathrm{phys}\rangle$

$$:(\Pi_\Psi\Psi)^{(P)}: |\mathrm{phys}\rangle = 0 \tag{2.45}$$

where : ... : stands for the normal ordering operator. The condition (2.45) states that the physical state vector is chargeless.

3 2D Heavy Quark QCD

3.1 Quantization of Heavy Quark QCD

Here we will consider the generalization of the results obtained in the previous Section to the case of non-Abelian gauge theory, in particular, to the two-dimensional heavy-quark QCD.

We start with the Lagrange density of $2D$ heavy-quark QCD

$$\mathcal{L} = i\sum_f \overline{\Psi}_f \not{U} U^\mu \mathcal{D}_\mu \Psi_f - M\sum_f \overline{\Psi}_f\Psi_f + \frac{1}{2}F^a_{+-}F^a_{+-} \tag{3.1}$$

where the index f stands for the flavor of the quark, Ψ_f is a color multiplet of quarks with a given flavor f, and it is assumed that the quark's mass is flavor independent,

$$\mathcal{D}_\mu = \partial_\mu + igA^a_\mu T^a \tag{3.2}$$

is the covariant derivative, T^a $(a = 1, \ldots, N_c^2 - 1)$ are the generators of Lie algebra corresponding to the group SU(N_c), obeying the following commutation relations

$$\left[T^a\ ,\ T^b\right]_- = if^{abc}T^c \tag{3.3}$$

with the structure constants f^{abc} being antisymmetric in all indices, and F^a_{+-} is the field strength tensor

$$\begin{aligned} F^a_{+-} &= \partial_+A^a_- - \partial_-A^a_+ + g\,(A_+\times A_-)^a\ , && (3.4)\\ A^a_\pm &= \frac{1}{\sqrt{2}}(A^a_0 \pm A^a_3) && (3.5)\end{aligned}$$

Here, for any two "isotopic" vectors, say, $\mathcal{F}^a$ and $\mathcal{G}^b$, the cross product is defined as

$$(\mathcal{F}\times\mathcal{G})^a = f^{abc}\mathcal{F}^b\mathcal{G}^c \tag{3.6}$$

For simplicity, in what follows, we will consider flavorless quarks.

The canonical momenta to the variables under consideration are

$$\begin{aligned} &\Pi_\Psi = i\overline{\Psi}\not{U}_-\ ,\ \ \Pi_{\overline{\Psi}} = 0\ ,\\ &\Pi^{a(P)}_- = P * \frac{\partial\mathcal{L}}{\partial\dot{A}_-} = \dot{A}^{a(P)}_- + g\,(A_+\times A_-)^{a(P)}\ ,\\ &\Pi^{a(Q)}_- = Q * \frac{\partial\mathcal{L}}{\partial\dot{A}_-} = \dot{A}^{a(Q)}_- - \partial_-A^{a(Q)}_+ + g\,(A_+\times A_-)^{a(Q)}\ ,\\ &\Pi^{a(P,Q)}_+ = 0 \end{aligned} \tag{3.7}$$

The velocities $\dot{A}_-^{a(P,Q)}$ can be expressed through the momenta $\Pi_-^{a(P,Q)}$, and one has the following primary constraints $\mathbf{\Phi} = 0$

$$\mathbf{\Phi} = \left\{ \begin{array}{l} \phi_+^{a(P,Q)} = \Pi_+^{a(P,Q)} \\ \phi_\Psi = \Pi_\Psi - i\overline{\Psi}\not{U}U_- \\ \phi_{\overline{\Psi}} = \Pi_{\overline{\Psi}} \end{array} \right. \tag{3.8}$$

On the surface of these constraints, the Lagrange density (3.1) in the Hamilton form reads as

$$\mathcal{L} = \Pi_-^{a(P)} \dot{A}_-^{a(P)} + \Pi_-^{a(Q)} \dot{A}_-^{a(Q)} + \Pi_\Psi \dot{\Psi} - \mathcal{H} \ , \tag{3.9}$$

where

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_F + \mathcal{H}_G, \\ \mathcal{H}_F &= -\frac{U_+}{U_-}\Pi_\Psi \partial_- \Psi - i\frac{M}{U_-}\Pi_\Psi \not{U}\Psi, \\ \mathcal{H}_G &= \frac{1}{2}\left(\Pi_-^{a(P)2} + \Pi_-^{a(Q)2}\right) + \Pi_-^{a(Q)}\partial_- A_+^{a(Q)} - g\Pi_-^{a(P)}\left(A_+ \times A_-\right)^{a(P)} \\ &\quad - g\Pi_-^{a(Q)}\left(A_+ \times A_-\right)^{a(Q)} - ig\frac{U_+}{U_-}\Pi_\Psi T^a \Psi \left(A_-^{a(P)} + A_-^{a(Q)}\right) \\ &\quad - ig\Pi_\Psi T^a \Psi \left(A_+^{a(P)} + A_+^{a(Q)}\right) \end{aligned} \tag{3.10}$$

Using the similar arguments from the previous Section, we neglect the term $\Pi_-^{a(P)}\Pi_-^{a(Q)}$ in the Hamilton density $\mathcal{H}_G$.

The sets of the initial variables $\zeta^{(0)}$ and the “generalized” momenta $a^{(0)}(\zeta)$ are

$$\begin{aligned} \zeta_j^{(0)} &= \left\{ A_-^{a(P)} \ , \ \Pi_-^{a(P)} \ , \ A_+^{a(P)} \ , \ A_-^{a(Q)} \ , \ \Pi_-^{a(Q)} \ , \ A_+^{a(Q)} \ , \ \Psi \ , \ \Pi_\Psi \right\} \ , \\ a_j^{(0)} &= \left\{ \zeta_2^{(0)} \ , \ 0 \ , \ 0 \ , \ \zeta_5^{(0)} \ , \ 0 \ , \ 0 \ , \ \zeta_8^{(0)} \ , \ 0 \ \right\} \quad , \quad j = 1, \ldots, 8 \end{aligned} \tag{3.11}$$

The corresponding simplectic supermatrix $f_{ij}^{(0)}$ has two eigenvectors with zero eigenvalues

$$\begin{aligned} V^{(P)} &= \{0 \ , \ 0 \ , \ c^a \ , \ldots \ 0\} \ , \\ V^{(Q)} &= \{0 \ , \ \ldots \ , \ c^a(x^-) \ , \ 0 \ , \ 0\} \end{aligned} \tag{3.12}$$

leading to two other sets of constraints

$$ig\left(\Pi_\Psi T^a \Psi\right)^{(P)} + g\left(A_- \times \Pi_-\right)^{a(P)} = 0 \ , \tag{3.13}$$

$$\partial_- \Pi_-^{a(Q)} + g\left(A_-^{(P)} \times \Pi_-^{(Q)}\right)^a + g\left(A_-^{(Q)} \times \Pi_-\right)^a + ig\Pi_\Psi T^a \Psi = 0 \tag{3.14}$$

Equation (3.13) tells that the total classical color charge in the system is zero.

The Hamilton density $\mathcal{H}_G$ on the constraint surface (3.8), (3.13), and (3.14) becomes

$$\mathcal{H}_G = \frac{1}{2}\left(\Pi_-^{a(P)}\right)^2 + \frac{1}{2}\left(\Pi_-^{a(Q)}\right)^2 - ig\frac{U_+}{U_-}\Pi_\Psi T^a \Psi (A_-^{a(P)} + A_-^{a(Q)}) \tag{3.15}$$

If one adds the constraints (3.13) and (3.14) with corresponding Lagrange multipliers to the Lagrange density (3.9), this does not give any new constraints. Therefore, one needs gauge conditions. Let us choose them, like in QED, in the form

$$\Omega_G^a = \partial_- A_-^{a(Q)} = 0 \tag{3.16}$$

Using the same arguments as in the previous Section, one obtains

$$A_-^{a(Q)} = 0, \quad \left(\text{note that } A_-^a \neq 0\right) \tag{3.17}$$

The Hamilton density $\mathcal{H}_G$ then can be sufficiently simplified

$$\mathcal{H}_G \quad = \quad \frac{1}{2}\left(\Pi_-^{a(P)}\right)^2 + \frac{1}{2}\left(\Pi_-^{a(Q)}\right)^2 - ig\frac{U_+}{U_-}\Pi_\Psi T^a \Psi A_-^{a(P)}$$

It can be shown, using the constraints (3.13), that the last term in the above expression does not give a contribution to the Hamiltonian H_G. Keeping this in mind, one can neglect it in the Hamilton density. Therefore, one can write

$$\mathcal{H}_G \quad = \quad \frac{1}{2}\left(\Pi_-^{a(P)}\right)^2 + \frac{1}{2}\left(\Pi_-^{a(Q)}\right)^2 \tag{3.18}$$

where the variables $\Pi_-^{a(Q)}$ have to be eliminated using the Gauss law constraints

$$\Omega_{GL}^a = \partial_- \Pi_-^{a(Q)} + g\left(A_-^{(P)} \times \Pi_-^{(Q)}\right)^a - g\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^a + ig\rho^a = 0 \tag{3.19}$$

where

$$\rho^a = \left(\Pi_\Psi T^a \Psi\right)^{(Q)} \tag{3.20}$$

In deriving (3.19), we have used the relation

$$(A_- \times \Pi_-)^{a(P)} = (A_-^{(P)} \times \Pi_-^{(P)})^a$$

which holds on the surface of the gauge condition (3.16) (or (3.17)).

Now the corresponding simplectic supermatrix $f_{ij}^{(2)}$ is nonsingular, and has a similar form as in 2D QED with

$$\begin{aligned} \{A_-^{a(P)}\ ,\ \Pi_-^{b(P)}\}_{FJ} \quad &= \quad \frac{1}{2L}\delta^{ab}\ , \\ \{\Pi_\Psi(x^-)\ ,\ \Psi(y^-)\}_{FJ} \quad &= \quad \delta(x^- - y^-) \end{aligned} \tag{3.21}$$

Therefore, one can consider the set of the variables

$$\omega^{\text{phys}} = \left\{(A_-^{a(P)},\ \Pi_-^{a(P)},\ \Pi_\Psi,\ \Psi\right\} \tag{3.22}$$

as physical ones. The quantization procedure consists of the replacement of the variables ω^{phys} by the corresponding operators

$$\omega^{\text{phys}} \to \hat{\omega}^{\text{phys}} \tag{3.23}$$

which obey the following commutation and anticommutation relations

$$[\hat{A}_-^{a(P)},\ \hat{\Pi}_-^{b(P)}]_- = \frac{i}{2L}\delta^{ab}, \tag{3.24}$$

$$[\hat{\Psi}(x^-),\ \hat{\Pi}_\Psi(y^-)]_+ = i\delta(x^- - y^-) \tag{3.25}$$

3.2 Physical Hamiltonian

To obtain the physical Hamiltonian, one has to solve the constraints (3.19) in order to express the momenta $\Pi_-^{a(Q)}$ in terms of physical variables. Although the eq.(3.19) is the one for the nonzero mode of the momenta $\Pi_-^{a(Q)}$, its left-hand-side contains a term which is a zero mode (the product of the zero modes does not have the nonzero mode). Thus, one has to be careful when such an equation has to be solved.

Let us define the function

$$Z^a(x^-) = V^{ab}(x^-)\Pi_-^{a(Q)}(x^-) \tag{3.26}$$

where

$$V^{ab}(x^-) = \left(\exp(gRx^-)\right)^{ab}, \quad \left(R^{ab} = f^{acb}A_-^{c(P)}\right) \tag{3.27}$$

is a unitary matrix. The Hamilton density (3.18) can be expressed through the function $Z^a(x^-)$

$$\mathcal{H}_G \;=\; \frac{1}{2}\left(\Pi_-^{a(P)}\right)^2 + \frac{1}{2}\left(Z^a(x^-)\right)^2 \tag{3.28}$$

From (3.19), the equation that the function $Z^a(x^-)$ satisfies is found to be

$$\partial_- Z^a(x^-) \;=\; \left(V(x^-)L(x^-)\right)^a, \tag{3.29}$$

$$L^a(x^-) \;=\; g\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^a - ig\rho^a(x^-) \tag{3.30}$$

The solution to the eq.(3.29) has the form

$$Z^a(x^-) = (VL)^{a(P)}\, x^- + \partial_-^{-1}\,(VL)^{a(Q)}\,(x^-) \tag{3.31}$$

We wish to stress out the existence of the linear term in (3.31). As we will see later, this term will give a contribution to the physical Hamiltonian where the momenta $\Pi_-^{a(P)}$ are involved.

To obtain the zero modes and nonzero modes of the expressions involved in (3.31), we remark that

$$L^{a(P)} \;=\; g\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^a \;\;, \;\; L^{a(Q)} = -ig\rho^a \tag{3.32}$$

Substituting these expressions into (3.26), one obtains the function $Z^a(x^-)$ in terms of the physical variables $\omega^{\rm phys}$ defined by (3.22)

$$\begin{aligned} Z^a(x^-) = g&\left\{V^{ac(P)}\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^c - iK^{a(P)}\right\}x^- \\ +g\partial_-^{-1}&\left\{(V^{ac(Q)}(x^-)\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^c - iK^{b(Q)} - iV^{bc(P)}\rho^c(x^-)\right\} \end{aligned} \tag{3.33}$$

where

$$K^a(x^-) = V^{ab(Q)}(x^-)\rho^b(x^-) \tag{3.34}$$

The eq.(3.33) makes possible to construct the physical Hamiltonian $H^{\rm phys}$

$$H^{\rm phys} \;=\; \int_{-L}^{L} \mathcal{H}^{\rm phys} dx^- \;, \tag{3.35}$$

$$\mathcal{H}^{\rm phys} \;=\; \mathcal{H}_F + \frac{1}{2}(\Pi_-^{a(P)})^2 + \frac{1}{2}(Z^a(x^-))^2 \tag{3.36}$$

where $Z^a(x^-)$ is given by (3.33) and can be presented in the following form

$$\begin{aligned} Z^a(x^-) &= gB^{a(P)}x^- + g\partial_-^{-1}B^{a(Q)}(x^-) \ , \\ B^{a(P)} &= G^{a(P)} - iK^{a(P)} \ , \\ B^{a(Q)}(x^-) &= G^{a(Q)}(x^-) - i\mathcal{P}^{a(Q)}(x^-) \\ G^a(x^-) &= V^{ab}(x^-)\left(A_-^{(P)} \times \Pi_-^{(P)}\right)^b = V^{ab}(x^-)R^{bc}\Pi_-^{c(P)} \ , \\ \mathcal{P}^{a(Q)}(x^-) &= K^{a(Q)}(x^-) + V^{ab(P)}\rho^b \end{aligned} \tag{3.37, 3.38}$$

Establishing the necessary tools, we can now consider the interaction Hamiltonian. Due to eqs.(3.36) and (3.38), the interaction Hamiltonian corresponds to the last term in (3.36)

$$H_{\rm int} = \frac{g^2}{2}\int_{-L}^{L}\mathcal{H}_{\rm int}(x)dx \tag{3.39}$$

where

$$\mathcal{H}_{\rm int}(x) = B^{a(P)}B^{a(P)}x^2 + 2xB^{a(P)}\partial^{-1}B^{a(Q)}(x) + \partial^{-1}B^{a(Q)}(x)\partial^{-1}B^{a(Q)}(x) \tag{3.40}$$

One can present the interaction Hamiltonian in the following form

$$H_{\rm int} = H_{\rm int}^{\rm zero} + H_{\rm int}^{\rm linear} + H_{\rm int}^{\rm quad} \tag{3.41}$$

where $H_{\rm int}^{\rm zero}$ is the part of interaction Hamiltonian which describes the self-interaction of the gauge zero modes in the sector of the pure Yang-Mills fields

$$\begin{aligned} H_{\rm int}^{\rm zero} &= \frac{g^2}{2}\left[\frac{2L^3}{3}G^{a(P)}G^{a(P)} + \int_{-L}^{L}\partial^{-1}G^{a(Q)}(x)\partial^{-1}G^{a(Q)}(x)dx \right. \\ &\left. -2G^{a(P)}\int_{-L}^{L}(\frac{x^2}{2} - \frac{L^2}{6})G^{a(Q)}(x)dx\right] \end{aligned} \tag{3.42}$$

the term $H_{\rm int}^{\rm linear}$ is the part of interaction Hamiltonian which is linear over the quark color charge ρ^a

$$\begin{aligned} H_{\rm int}^{\rm linear} &= -ig^2\left[\frac{2L^3}{3}G^{a(P)}K^{a(P)} + \int_{-L}^{L}\partial^{-1}G^{a(Q)}(x)\partial^{-1}\mathcal{P}^{a(Q)}(x)dx \right. \\ &\left. -\int_{-L}^{L}(\frac{x^2}{2} - \frac{L^2}{6})\left(G^{a(P)}\mathcal{P}^{a(Q)}(x) + K^{a(P)}G^{a(Q)}(x)\right)dx\right] \end{aligned} \tag{3.43}$$

and the term $H_{\rm int}^{\rm quad}$ describes the quark-quark interaction

$$\begin{aligned} H_{\rm int}^{\rm quad} &= -\frac{g^2}{2}\left[\frac{2L^3}{3}K^{a(P)}K^{a(P)} + \int_{-L}^{L}\partial^{-1}\mathcal{P}^{a(Q)}(x)\partial^{-1}\mathcal{P}^{a(Q)}(x)dx \right. \\ &\left. -2K^{a(P)}\int_{-L}^{L}(\frac{x^2}{2} - \frac{L^2}{6})\mathcal{P}^{a(Q)}(x)dx\right] \end{aligned} \tag{3.44}$$

In deriving these expressions, we used the following properties of the operator ∂^{-1}

$$\begin{aligned} \int_{-L}^{L}(\partial^{-1}f_1^{(Q)})(x)f_2^{(Q)}(x)dx &= -\int_{-L}^{L}(\partial^{-1}f_2^{(Q)})(x)f_1^{(Q)}(x)dx \ , \\ \partial^{-1}x &= \frac{x^2}{2} - \frac{L^2}{6} \end{aligned} \tag{3.45}$$

For the Abelian case, the only term that survives in (3.41) is $H_{\rm int}^{\rm quad}$, given by (3.44) with $K^a = 0$ and $\mathcal{P}^{(Q)}(x) = (\Pi_\Psi \Psi)^{(Q)}$. This gives the result obtained in the previous Section.

Let us now return to the constraint (3.13). Using the gauge condition (3.16) (or (3.17)), it can be rewritten as

$$i\,(\Pi_\Psi T^a \Psi)^{(P)} + \left(A_-^{(P)} \times \Pi_-^{(P)}\right)^a = 0 \tag{3.46}$$

This constraint is of the first class. We do not know how to fix the gauge corresponding to this constraint. Therefore, as in the previous Section, we will consider the constraint (3.46) as a strong one, meaning that it should be satisfied on the physical state vectors $|\mathrm{phys}\rangle$

$$:\left(i\,(\Pi_\Psi T^a \Psi)^{(P)} + \left(A_-^{(P)} \times \Pi_-^{(P)}\right)^a\right): |\mathrm{phys}\rangle = 0 \tag{3.47}$$

where : ... : stands for the normal ordering operator.

3.3 The group SU(2)

Let us examine the results obtained for the SU(2) group. In this case, the structure constants f^{abc} coincide with the antisymmetric Levi-Civita tensor ϵ^{abc}

$$f^{abc} = \epsilon^{abc} \quad , \quad a = 1,2,3, \quad (\epsilon^{123} = 1) \tag{3.48}$$

The matrix $V^{ab}(x^-)$ can be calculated and is found to be

$$\begin{aligned} V^{ab}(x^-) &= \delta^{ab} + V_1(x^-)\frac{(R^2)^{ab}}{D^2} + V_2(x^-)\frac{R^{ab}}{D} \ , \\ V_1(x^-) &= 1 - \cos\left(gx^- D\right) \quad , \quad V_2(x^-) = \sin\left(gx^- D\right) \end{aligned} \tag{3.49}$$

where

$$D = \sqrt{A_-^{a(P)} A_-^{a(P)}} \quad , \quad \left(R^2\right)^{ab} = A_-^{a(P)} A_-^{b(P)} - \delta^{ab} D^2 \tag{3.50}$$

From these expressions, one can explicitly find the zero modes and nonzero modes of the matrix $V^{ab}(x^-)$ involved in the physical Hamiltonian

$$\begin{aligned} V^{ab(Q)}(x^-) &= u_1(x^-)\frac{(R^2)^{ab}}{D^2} + u_2(x^-)\frac{R^{ab}}{D} \ , \\ V^{ab(P)} &= \delta^{ab} - u_1(0)\frac{(R^2)^{ab}}{D^2} \ , \\ u_1(x^-) &= \frac{\sin(gDL)}{gDL} - \cos(gDx^-) \quad , \quad u_2(x^-) = \sin(gDx^-) \end{aligned} \tag{3.51}$$

Other quantities that are involved in the problem are

$$\begin{aligned} \partial_-^{-1} V^{ab(Q)}(x^-) &= v_1(x^-)\frac{(R^2)^{ab}}{D^2} + v_2(x^-)\frac{R^{ab}}{D} \ , \\ v_1(x^-) &= -\frac{1}{gD}u_2(x^-) + x^-\frac{\sin(gDL)}{gDL} \quad , \quad v_2(x^-) = \frac{1}{gD}u_1(x^-) \ , \\ \partial_-^{-2} V^{ab(Q)}(x^-) &= z_1(x^-)\frac{(R^2)^{ab}}{D^2} + z_2(x^-)\frac{R^{ab}}{D} \ , \end{aligned} \tag{3.52}$$

$$
\begin{aligned}
z_1(x^-) &= -\frac{1}{g^2D^2}u_1(x^-) + (\frac{(x^-)^2}{2} - \frac{L^2}{6})\frac{\sin(gDL)}{gDL} \;, \\
z_2(x^-) &= -\frac{1}{g^2D^2}u_2(x^-) + \frac{x^-}{gD}\frac{\sin(gDL)}{gDL} \;, \\
G^{a(P)} &= \frac{\sin gDL}{gDL}R^{ab}\Pi_-^{b(P)} \;, \\
G^{a(Q)}(x^-) &= -\; D\left(u_1(x^-)\frac{R^{ab}}{D} - u_2(x^-)\frac{(R^2)^{ab}}{D^2}\right)\Pi_-^{b(P)} \;,
\end{aligned}
\tag{3.53}
$$

$$
K^a(x^-) = \left(u_1(x^-)\frac{(R^2)^{ab}}{D^2} + u_2(x^-)\frac{R^{ab}}{D}\right)\rho^b(x^-) \tag{3.54}
$$

The part $H_{\rm int}^{\rm zero}$ of the interaction Hamiltonian can be simplified and presented in the following form

$$
H_{\rm int}^{\rm zero} = -L\left[1 - \left(\frac{\sin(gDL)}{gDL}\right)^2\right]\Pi_-^{a(P)}\frac{(R^2)^{ab}}{D^2}\Pi_-^{b(P)} \tag{3.55}
$$

This term describes the self-interaction of the gluons in terms of their zero modes.

The result for the part of the interaction Hamiltonian $H_{\rm int}^{\rm linear}$ is

$$
H_{\rm int}^{\rm linear} = -ig^2D\left(\int_{-L}^{L}\lambda_1(x)\Pi_-^{a(P)}\frac{(R^2)^{ab}}{D^2}\rho^b(x)dx + \int_{-L}^{L}\lambda_2(x)\Pi_-^{a(P)}\frac{R^{ab}}{D}\rho^b(x)dx\right) \tag{3.56}
$$

where

$$
\begin{aligned}
\lambda_1(x) &= -\frac{x}{gD}\frac{\sin(gDL)}{gDL}\cos(gDx) + \frac{\cos(gDL)}{g^2D^2}\sin(gDx), \\
\lambda_2(x) &= \frac{x}{gD}\frac{\sin(gDL)}{gDL}\sin(gDx) - \frac{1}{g^2D^2}\cos(gDL)\left(\frac{\sin(gDL)}{gDL} - \cos(gDx)\right)
\end{aligned}
\tag{3.57}
$$

This term describes the interaction of the heavy-quarks with the zero modes of the gauge degrees of freedom.

The part $H_{\rm int}^{\rm quad}$ of the interaction Hamiltonian, which describes the self-interaction between heavy-quarks, has the following form

$$
\begin{aligned}
H_{\rm int}^{\rm quad} &= g^2\int_{-L}^{L}\int_{-L}^{L}\left(A_1(x,y)\rho^a(x)\frac{(R^2)^{ab}}{D^2}\rho^b(y) + A_2(x,y)\rho^a(x)\frac{R^{ab}}{D}\rho^b(y)\right)dxdy \\
&+g^2\int_{-L}^{L}\int_{-L}^{L}\rho^a(x)\left(\delta^{ab} + \frac{(R^2)^{ab}}{D^2}\right)\rho^b(y)H^{(Q)}(x-y)dxdy
\end{aligned}
\tag{3.58}
$$

where

$$
\begin{aligned}
A_1(x,y) &= \frac{L}{6}\left(u_1(x)u_1(y) + u_2(x)u_2(y)\right) \\
&+\frac{1}{2L}\left\{\left[(\frac{x^2}{2} - \frac{L^2}{6})\left(u_1(x)\cos(gDy) - u_2(x)\sin(gDy)\right)\right] + [x \leftrightarrow y]\right\} \\
&+\frac{1}{2}\left[\left(u_1(x)\cos(gDy) - u_2(x)\sin(gDy)\right) + (x \leftrightarrow y)\right]H^{(Q)}(x-y) \\
&-\frac{\sin(gDL)}{2gDL}\left(\cos(gDx) + \cos(gDy)\right)H^{(Q)}(x-y),
\end{aligned}
\tag{3.59}
$$

$$
\begin{aligned}
A_2(x,y) &= \frac{L}{6}\left(u_1(x)u_2(y)-u_2(x)u_1(y)\right)\\
&-\frac{1}{2L}\left\{\left[\left(\frac{x^2}{2}-\frac{L^2}{6}\right)\left(u_2(x)\cos(gDy)+u_1(x)\sin(gDy)\right)\right]-[x\leftrightarrow y]\right\}\\
&-\frac{1}{2}\left[\left(u_2(x)\cos(gDy)+u_1(x)\sin(gDy)\right)-(x\leftrightarrow y)\right]H^{(Q)}(x-y)\\
&+\frac{\sin(gDL)}{2gDL}\left(\sin(gDx)-\sin(gDy)\right)H^{(Q)}(x-y)
\end{aligned}
\tag{3.60}
$$

where $H^{(Q)}(x-y)$ is given by (2.39).

From (3.36) and (3.55), one can find the total Hamiltonian corresponding to the pure Yang-Mills fields

$$
H_{\text{pure YM}} = L\Pi_-^{a(P)}G^{ab}\Pi_-^{b(P)} \tag{3.61}
$$

where

$$
G^{ab} = \delta^{ab} - \left(1-\left(\frac{\sin gDL}{gDL}\right)^2\right)\frac{(R^2)^{ab}}{D^2} \tag{3.62}
$$

The Hamiltonian $H_{\text{pure YM}}$ is quadratic in the momenta $\Pi_-^{a(P)}$ and sufficiently nonlinear in the coordinates $A_-^{a(P)}$.

3.4 The Faddeev-Popov Determinant

The constraints (3.19) and gauge conditions (3.16) are the only local constraints in the problem. Although these constraints are linear over the variables $\Pi_-^{a(Q)}$ and $A_-^{a(Q)}$, their Poisson bracket is nontrivial and depends on the physical variables $A_-^{a(P)}$. This might give a nonzero contribution to the effective interaction potential. To find such a contribution, we should calculate the Faddeev-Popov determinant which appears due to the constraints (3.16) and (3.19). One should introduce the Faddeev-Popov ghosts provided that the Faddeev-Popov determinant is not trivial (it may depend on the field variables). On the other hand, the Faddeev-Popov determinant is important for the unitarity of the S-matrix.

In order to find the Faddeev-Popov determinant, one should consider the Poisson bracket between the first-class constraints (3.19) and the corresponding gauge conditions (3.16). The result is found to be

$$
\begin{aligned}
\{\Omega_G^a(x)\ ,\ \Omega_{GL}^b(y)\}_{PB} &= -\partial_x\mathcal{D}_-^{ab}(x)D^{(Q)}(x-y)\ , &(3.63)\\
\mathcal{D}_-^{ab}(x) &= \delta^{ab}\partial_x + gR^{ab},\ \ R^{ab}=f^{acb}A_-^{c(P)} &(3.64)
\end{aligned}
$$

where

$$
D^{(Q)}(x-y) = \delta(x-y) - \frac{1}{2L} \tag{3.65}
$$

is the δ-function in the Q sector. (For simplicity, we omitted the index "-" in the coordinates x and y.)

The Faddeev-Popov determinant is defined as

$$
\Delta[A] = \mathrm{Det}^{1/2}\{\Psi_\ell,\Psi_{\ell'}\}_{PB}|_{\Psi=0} \tag{3.66}
$$

where Ψ_ℓ is the set of first class constraints and the corresponding gauge conditions. In our case, the Faddeev-Popov determinant depends only on the variables $A_-^{(P)}$ and can be presented as

$$\Delta[A_-^{(P)}] \quad = \quad \mathrm{Det}\left(\frac{\partial_-\mathcal{D}_-}{\partial_-^2}\right) = \mathrm{Det}\left(\frac{\mathcal{D}_-}{\partial_-}\right) \tag{3.67}$$

In order to normalize the Faddeev-Popov determinant, we have introduced the operator ∂_-^2 in the denominator of (3.67). Using the well known property of the determinant of a matrix

$$\mathrm{Det}M = \exp\left(\mathrm{Tr}\ \mathrm{Ln}M\right)$$

one obtains

$$\Delta[A_-^{(P)}] = \exp\left(\mathrm{Tr}\ \mathrm{Ln}\frac{\mathcal{D}_-}{\partial_-}\right) \tag{3.68}$$

Taking the derivative of the both sides of (3.68) with respect to the coupling constant g, one obtains an ordinary differential equation for the Faddeev-Popov determinant

$$\frac{d\Delta[A_-^{a(P)}]}{dg} \quad = \quad \left(\mathrm{Tr}\mathcal{D}_-^{-1}(g)R\right)\Delta[A_-^{a(P)}] = \left(\mathrm{Tr}G(|g)R\right)\Delta[A_-^{a(P)}] \tag{3.69}$$

where $G^{ab}(x,y|g)$ is the Green function of the operator $\mathcal{D}_-^{ab}$. The solution of this equation has the following form

$$\Delta[A_-^{a(P)}] = \exp\left(\int_0^g dg'\mathrm{Tr}G(|g')R\right) \tag{3.70}$$

The determinant (3.67) is equivalent to the Faddeev-Popov term in the effective action

$$S_{FP} = \int dx^+dx^-\bar{c}^a(x)\partial_-\mathcal{D}_-^{ab}(x)c^b(x) \tag{3.71}$$

where c^b and $\bar{c}^a$ are the ghost and antighost fields, respectively. The operator $\mathcal{D}_-^{ab}(x)$ depends only on the zero modes $A_-^{a(P)}$. Therefore, only the nonzero modes of the ghost and antighost fields give contributions to the Faddeev-Popov action, meaning that the Faddeev-Popov ghosts are necessary only in their own Q-sector. One should then make the following substitution

$$\bar{c}^a \rightarrow \bar{c}^{a(Q)}, \quad c^a \rightarrow c^{a(Q)}$$

and the Faddeev-Popov action becomes

$$S_{FP} = \int dx^+dx^-\bar{c}^{a(Q)}(x)\partial_-\mathcal{D}_-^{ab}(x)c^{b(Q)}(x) \tag{3.72}$$

where $\bar{c}^{a(Q)}$ and $c^{a(Q)}$ are the nonzero modes of the Faddeev-Popov ghosts.

Let us compute the Faddeev-Popov determinant. As it follows from (3.70), one needs to know the Green function $G^{ab}(x,y|g)$. According to (3.72), it satisfies the equation

$$\partial_x G^{ab}(x,y|g) + gR^{ac}G^{cb}(x,y|g) = \delta^{ab}D^{(Q)}(x-y) \tag{3.73}$$

The matrix R^{ab} does not depend on x. Consequently, the Green function $G^{ab}(x,y|g)$ is translational invariant: $G^{ab}(x,y|g) = G^{ab}(x-y|g) = G^{ab}(z|g)$, and (3.71) can be rewritten as

$$\partial_-^z G^{ab}(z|g) + gR^{ac}G^{cb}(z|g) = \delta^{ab}D^{(Q)}(z) \tag{3.74}$$

This equation is an inhomogeneous first order differential equation, and its solution can be presented in the following form

$$G^{ab}(z|g) = U^{ac}(z|g)\left(V^{cb}(|g)D^{(Q)}\right)^{(P)} z + U^{ac}(z|g)\partial^{-1}\left(V^{cb}(|g)D^{(Q)}\right)^{(Q)}(z) \tag{3.75}$$

where the matrix $V^{ab}(z|g)$ was defined by (3.27), and the matrix $U^{ab}(z|g)$ is its inverse one. Substituting (3.75) into (3.70), one obtains

$$\Delta[A_-^{a(P)}] = \exp\left(2L\int_0^g dg'\partial^{-1}\left(V^{ab}(|g')D^{(Q)}\right)^{(Q)}(z)|_{z=0}R^{ba}\right) \tag{3.76}$$

The integrand in the right-hand-side of eq. (3.76) can be presented as

$$\begin{aligned}&\partial^{-1}\left(V^{ab}(|g')D^{(Q)}\right)^{(Q)}(z) = \\ &= V^{ab(P)}(g')G^{(Q)}(z) + \partial^{-1}\left(V^{ab(Q)}(|g')D^{(Q)}\right)^{(Q)}(z)\end{aligned} \tag{3.77}$$

where the function $G^{(Q)}(z) = \frac{\epsilon(z)}{2} - \frac{z}{2L}$ was defined by (2.26). Thus

$$\Delta[A_-^{a(P)}] = \exp\left(2L\int_0^g dg'\partial^{-1}\left(V^{ab(Q)}(|g')D^{(Q)}\right)^{(Q)}(z)|_{z=0}R^{ba}\right) \tag{3.78}$$

This is, so far, the most general result we can obtain for the Faddeev-Popov determinant. We cannot go further without specifying the gauge group. Therefore, for simplicity, in what follows we will consider the group SU(2). Using the representation (3.51) for the matrix $V^{ab(Q)}$ and the definition of the operator ∂^{-1}, one obtains

$$\Delta[A_-^{a(P)}] = \exp\left(-4DL\int_0^g dg'\partial^{-1}\left(u_2(|g')D^{(Q)}\right)^{(Q)}(z)|_{z=0}\right), \tag{3.79}$$

$$\partial^{-1}\left(u_2(|g')D^{(Q)}(z)\right)^{(Q)}|_{z=0} = \frac{1}{2g'DL}\left(1 - \frac{\sin(g'DL)}{g'DL}\right) \tag{3.80}$$

Thus

$$\Delta[A_-^{a(P)}] = \exp\left\{-2\int_0^{gDL} x^{-1}\left(1 - \frac{\sin x}{x}\right)dx\right\} \tag{3.81}$$

The Faddeev-Popov determinant depends on the zero modes $A_-^{a(P)}$ through the upper limit in the integral involved in (3.81). This means that one does need to introduce the Faddeev-Popov ghosts (at least for the group SU(2)), and there is an additional term to the self-interaction Hamiltonian of the gluons.

The integral that appears in the right-hand-side of (3.81) can be presented in the following form

$$\int_0^z x^{-1}\left(1 - \frac{\sin x}{x}\right)dx = -\left(1 - \frac{\sin z}{z}\right) + (Ci(z) - \gamma - \ln z) \tag{3.82}$$

where $Ci(z)$ is the cosine integral special function, and γ is the Euler constant. Therefore

$$\Delta[A_-^{a(P)}] \;=\; \exp\left\{2\left(1 - \frac{\sin(gDL)}{gDL}\right) - 2\left(Ci(gDL) - \gamma - \ln(gDL)\right)\right\} \tag{3.83}$$

4 Conclusions

We have considered the light-cone quantization of the two-dimensional heavy-quark QCD, explicitly taking into account the zero modes contribution of the gauge degrees of freedom. We have imposed the periodic boundary conditions for the gauge degrees of freedom and the antiperiodic ones for the fermions. As ordinary QCD, this model is gauge invariant, meaning that there are unphysical degrees of freedom in the problem. We have used the Faddeev-Jackiw algorithm to quantize the theory . In order to see the role of the zero modes explicitly, we have considered the gauge conditions which are needed only in the nonzero mode sector. We obtained the physical variables (coordinates and their conjugate momenta) and the corresponding (anti-) commutation relations. We found that the physical variables are the zero modes of the "spatial" light-cone gauge degrees of freedom $A_-^{a(P)}$ and $\Pi_-^{a(P)}$, and the fermionic variables Ψ and Π_Ψ. Solving the constrains and the gauge conditions in order to eliminate the unphysical gauge degrees of freedom, we constructed the physical Hamiltonian. In the elimination of the unphysical variables mentioned above, we found that the expression for the momenta $\Pi_-^{a(Q)}$ contains an additional term that is linear in x^- (see (3.31)). Such a term gives a contribution to both the self-interaction and mutual-interaction potentials of the gluon fields and quark fields. We found that one needs to introduce the Faddeev-Popov ghosts in their own nonzero mode sector. The Faddeev-Popov determinant was calculated, and it was found that it depends on the zero modes $A_-^{a(P)}$, giving a contribution to the self-interaction Hamiltonian of the gluons.

ACKNOWLEDGMENT

It is pleasant to thank R.W. Brown, V.K. Mishra and C.C. Taylor for useful discussions and remarks that greatly improved the manuscript. Sh.Sh. is also grateful to E.S. Fradkin for reading the manuscript and providing valuable remarks.

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